Binomial Theorem

Unless otherwise stated, c_r denotes the coefficient of the x^r term in the expansion of $(1 + x)^n$.

1. Use the binomial theorem to find the exact value of $(10.1)^5$.

2. Use the binomial theorem to evaluate $\left(2+\sqrt{3}\right)^4 + \left(2-\sqrt{3}\right)^4$.

Hence, without using tables, show that $\left(2+\sqrt{3}\right)^4$ lies between 193 and 194.

3. **(a)** Find the values of the constants a and b if the expansion of $(1 + ax + bx^2)^6$ in ascending powers of x as far as the term x^2 is $1 - 12x + 78x^2$.

(b) Find the value of the term independent of x in the expansion of $\left(3x^2 - \dfrac{1}{2x}\right)^9$.

4. Find the greatest term in the expansion of:

(a) $(1 + 4x)^8$ when $x = \dfrac{1}{3}$.

(b) $(3 + 2x)^9$ when $x = 1$.

5. Prove that the coefficient of x^n in the expansion of $(1 + x)^{2n}$ is double the coefficients of x^n in the expansion of $(1 + x)^{2n-1}$.

6. Show that the coefficients of x^m and x^n in $(1 + x)^{m+n}$ are equal.

7. Show that for one value of r the coefficient of x^r in the expansion of $(3 + 2x - x^2)(1 + x)^{34}$ is zero.

8. Show that, if n is even, the coefficient of the middle term of $(1 + x)^n$ is

$\dfrac{1\cdot3\cdot5\cdots(n-1)}{1\cdot2\cdot3\cdots\left(\dfrac{n}{2}\right)}2^{\frac{n}{2}}$; and that, if n is odd, the coefficient of each of the two middle terms is $\dfrac{1\cdot3\cdot5\cdots n}{1\cdot2\cdot3\cdots\left(\dfrac{n+1}{2}\right)}2^{\frac{n-1}{2}}$.

9. Prove that : $c_1 - 2c_2 + 3c_3 - \ldots + n(-1)^{n-1} c_n = 0$.

10. Prove that : $c_0c_r + c_1c_{r+1} + c_2c_{r+2} + \ldots + c_{n-r}c_n = \dfrac{(2n)(2n-1)....(n-r+1)}{(n+r)!}$.

11. Prove that : $\displaystyle\sum_{r=0}^{n-1} c_r c_{r+1} = \frac{2(2n-1)!}{(n+1)[(n-1)!]^2}$.

12. Prove that : $\displaystyle\sum_{r=0}^{n-1} rc_{r+1} = 1 + (n-2)2^{n-1}$.

13. If ${}_nC_r$ is the coefficient of x^r in the binomial expansion of $(1 + x)^n$, prove that:

(a) ${}_nC_r = {}_{n+1}C_r - {}_nC_{r-1}$.

(b) $(3n + 1)\,[{}_{2n}C_n]^2 = (n + 1)\,\{[{}_{2n+1}C_n]^2 - [{}_{2n}C_{n-1}]^2\}$

14. If $(1 + x)^{2n} = c_0 + c_1x + c_2x^2 + \ldots + c_{2n}x^{2n}$, show that $c_0 + c_2 + c_4 + \ldots + c_{2n} = 2^{2n-1}$.

15. Show that if $(1 + x + x^2)^{10} = c_0 + c_1x + c_2x^2 + \ldots + c_{20}x^{20}$, then

$c_1 + c_2 + c_3 + \ldots + c_{19} = c_2 + c_4 + \ldots + c_{20}$.

16. If $(1 + x)^{2m+1} = c_0 + c_1x + c_2x^2 + \ldots + c_{2m+1}x^{2m+1}$, where m is a positive integer, show that:

$c_0 + c_1 + c_2 + \ldots + c_m = 2^{2m}$.

17. If $x^n = p_0(x - a)^n + p_1(x - a)^{n-1} + p_2(x - a)^{n-2} + \ldots + p_n$, show that $p_r = ({}_nC_r)\, a^r$.

18. Prove that: ${}_{n}C_{n} + {}_{n+1}C_{n} + {}_{n+2}C_{n} + \ldots + {}_{n+k}C_{n} = {}_{n+k+1}C_{n+1}$.

(Hint: Consider $(1 + x)^n + (1 + x)^{n+1} + \ldots + (1 + x)^{n+k}$.)

19. In the expansion of $(1 + x)^n$, where n is a positive integer, by the binomial theorem, put $x = 1$ and hence show that $(n - 1)!\,(2^n - 2)$ is divisible by n.

Hence deduce Fermat's theorem that if n is any prime number, $2^{n-1} - 1$ is divisible by n.

20. If $f(r) = c_0c_r + c_1c_{r-1} + c_2c_{r-2} + \ldots + c_rc_0$, where c_r denotes the coefficient of x^r in the expansion of $(1 + x)^n$, prove that:

(a) $f(r) = \dfrac{(2n)!}{r!(2n-r)!}$,

(b) $c_0 f(1) + 2c_1 f(2) + 3c_2 f(3) + \ldots + (n + 1)c_n f(n + 1) = \dfrac{2(3n-1)!}{(2n-1)!(n-1)!}$

21. Prove the following identities:

(a) $({}_{n}C_0)^2 + ({}_{n}C_1)^2 + ({}_{n}C_2)^2 + \ldots + ({}_{n}C_n)^2 = {}_{2n}C_n$,

(b) $({}_{2n}C_0)^2 - ({}_{2n}C_1)^2 + ({}_{2n}C_2)^2 + \ldots - ({}_{2n}C_{2n})^2 = (-1)^n\, {}_{2n}C_n$,

(c) $({}_{2n+1}C_0)^2 - ({}_{2n+1}C_1)^2 + ({}_{2n+1}C_2)^2 + \ldots - ({}_{2n+1}C_{2n+1})^2 = 0$,

(d) $({}_{n}C_1)^2 + 2({}_{n}C_2)^2 + \ldots + n({}_{n}C_n)^2 = \dfrac{(2n-1)!}{[(n-1)!]^2}$.

22. Prove that the sum of $n + 1$ terms of the series: $a + n(a + b) + {}_{n}C_2(a + 2b) + {}_{n}C_3(a + 3b) + \ldots$ is $2^{n-1}(2a + nb)$.

23. **(a)** Show that: $(1 + x)^{2n} + x(1 + x)^{2n-1} + \ldots + x^n (1 + x)^n + x^{n+1}(1 + x)^{n-1} + \ldots + x^{2n}$

is equal to $(1 + x)^{2n+1} - x^{2n+1}$.

(b) By using (a), or otherwise, show that the coefficient of x^n term in

$(1 + x)^{2n} + x(1 + x)^{2n-1} + \ldots + x^n (1 + x)^n$ is equal to ${}_{2n+1}C_n$.

24. Prove that: ${}_{n}C_1\, 1^2 + {}_{n}C_2\, 2^2 + \ldots + {}_{n}C_n\, n^2 = n(n + 1)\, 2^{n-2}$. (Hint: ${}_{n}C_k\, k^2 = k(k - 1)\, {}_{n}C_k + k\, {}_{n}C_k$.)

25. In the expansion of $(1 + x + x^2)^n$, the coefficient of x^r term is a_r. Prove that:

(a) $a_0 + a_1 + \ldots + a_{2n} = 3^n$,

(b) $a_0 - a_1 + a_2 - a_3 + \ldots + a_{2n} = 1$,

(c) $a_{n-r} = a_{n+r}$,

(d) $a_0^2 - a_1^2 + a_2^2 - a_3^2 + \ldots + a_{2n}^2 = a_n$,

(e) $a_0^2 - a_1^2 + a_2^2 - a_3^2 + \ldots + (-1)^{n-1} a_{n-1}^2 = \dfrac{1}{2}a_n\left[1-(-1)^n a_n\right]$

(f) $a_0a_2 - a_1a_3 + a_2a_4 - a_3a_5 + \ldots + a_{2n-2}a_{2n} = a_{n+1}$.

26. **(a)** Write down the formula for the sum of the coefficients in the expansion of $(1 + x)^m$, where m is a positive integer.

(b) Deduce $\dfrac{1}{1!(2n)!} + \dfrac{1}{2!(2n-1)!} + \dfrac{1}{3!(2n-2)!} + \ldots + \dfrac{1}{n!(n+1)!} = \dfrac{2^{2n}-1}{(2n+1)!}$.

27. If n is a positive integer and ${}_{n}C_{18} = {}_{n}C_7$, find the values of ${}_{n}C_{22}$ and ${}_{27}C_n$.

28. If a_r is the coefficient of x_r term in the expansion of $(1 + x)^n$, prove that:

$$\frac{a_1}{a_0} + 2\frac{a_2}{a_1} + 3\frac{a_3}{a_2} + \ldots + n\frac{a_n}{a_{n-1}} = \frac{1}{2}n(n+1) .$$

29. Find a_r, the coefficient of x^r, in the expansion of: $(1 + x)^n + (1 + 2x)^n + (1 + 4x)^n$, where n is a positive integer. Find the ratio $a_3 : a_{n-3}$ when $n = 9$.

30. If c_r is the coefficient of x_r in the expansion of $(1 + x)^n$,

where n is a positive integer, show that $c_0 + \frac{c_1}{2} + \frac{c_2}{3} + \ldots + \frac{c_n}{n+1} = \frac{2^{n+1}-1}{n+1}$.

31. If $(1 + x)^{2n} = c_0 + c_1x + c_2x^2 + \ldots + c_{2n}x^{2n}$, show that $2^{2n-1} = c_0 + c_2 + \ldots + c_{2n}$.

Sum the series: $1+(2p-1)x+(3p-2)(p-1)\frac{x^2}{2!}+(4p-3)\frac{(p-1)(p-2)}{3!}x^3+\ldots$, where p is a positive integer.

32. If n is a positive integer, prove that the coefficients of x^2 and x^3 in the expansion of $(x^2 + 2x + 2)^n$ are $2^{n-1}n^2$ and $\frac{1}{3}2^{n-1}n(n^2-1)$ respectively.

33. Find the coefficients of the terms in x^5 and $1/x^5$ in the expansion of $\left(1+x-\frac{2}{x^3}\right)^7$ in powers of x.

34. If $(1 - 2x + 2x^2)^{10} = 1 + ax + bx^2 + \ldots$, find the values of a and b.

35. Find the positive integral value of n which makes the ratio of the coefficient x^4 to that of x^3 in the expansion of $(1 + 2x + 3x^2)^n$ in a series of powers of x is equal to 121/28.

36. If $(1 + x + x^2)^{3n} = c_0 + c_1x + c_2x^2 + \ldots + c_{6n}x^{6n}$, prove that $c_0 - c_1 + c_2 - c_3 + c_4 - \ldots + c_{6n} = 1$.

37. If, in the expansion of $(a + x)^n$, s_1 is the sum of the odd terms and s_2 is the sum of the even terms, show that: $s_1^2 - s_2^2 = (a^2 - x^2)^n$, and $4s_1s_2 = (a + x)^{2n} - (a - x)^{2n}$.

38. By comparing the coefficients of x^r on both sides of the identity $(1 + x)^n = (1 + x)^2 (1 + x)^{n-2}$, prove that: ${}_nC_r = {}_{n-1}C_r + 2({}_{n-2}C_{r-1}) + {}_{n-2}C_{r-2}$.

39. If $c_r = {}_nC_r$, where $r = 0, 1, 2, 3, \ldots, n$., and if $f(r) = c_0c_r + c_1c_{r-1} + c_2c_{r-2} + \ldots + c_rc_0$, prove that $f(r) = {}_{2n}C_r$.

40. Show that for one value of r the coefficient of x^r in the expansion of $(3 + 2x - x^2)(1+x)^{34}$ is zero.

41. If $(1 - x + x^2)^{3n} = a_0 + a_1x + a_2 x^2 + \ldots$ and $(x + 1)^{3n} = c_0x^{3n} + c_1x^{3n-1} + c_2x^{3n-2} + \ldots$

prove that $a_0 + a_1 + a_2 + \ldots = 1$ and $a_0c_0 + a_1c_1 + a_2c_2 + \ldots = \frac{(3n)!}{n!(2n)!}$

42. Find a positive integer p such that the coefficients of x and x^2 in the expansion of $(1 + x)^{2p}(1 - x)^p$ are equal.

43. If I is the integral part and F the fractional part of $(3\sqrt{3}+5)^{2n+1}$, prove that $F(I + F) = 2^{2n+1}$.

44. Given positive integers m, n, let $\sum_{r=0}^{mn} a_r x^r \equiv (1+x+\ldots+x^m)^n$ (*)

(a) Find expressions for the following in terms of m an n only:

(i) $\sum_{r=0}^{mn} a_r$ **(ii)** $\sum_{r=0}^{mn} (-1)^r a_r$ **(iii)** $\sum_{r=0}^{mn} 2^r a_r$

(b) By differentiating (*), or otherwise, show that $mn\sum_{r=0}^{mn} a_r = 2\sum_{r=0}^{mn} ra_r$.

(c) Show that: $\sum_{r=1}^{m} (-1)^r r = \begin{cases} \frac{m}{2} & ,\text{if } m \text{ is even} \\ -\frac{m+1}{2} & ,\text{if } m \text{ is odd.} \end{cases}$ Hence, show that $2\sum_{r=1}^{m} (-1)^r ra_r = \begin{cases} mn & ,\text{if } m \text{ is even} \\ 0 & ,\text{if } m \text{ is odd.} \end{cases}$

45. Prove that $\quad {}_{n}C_{r+1} = {}_{n-1}C_{r} + {}_{n-1}C_{r+1}$. Hence show that for $n > r$, $\quad {}_{n}C_{r} = \sum_{i=0}^{r} {}_{n-i-1}C_{r-i}$.

46. Show that $\quad {}_{n}C_{r} = {}_{n-1}C_{r-1} + {}_{n-1}C_{r}$. Hence, by induction or otherwise, evaluate $\sum_{q=0}^{n} {}_{n+q}C_{q} \dfrac{1}{2^{n+q}}$.

47. Polynomials $C_r(x)$ are defined by $\quad C_0(x) = 1$ and $C_r(x) = \dfrac{x(x-1)\dots(x-r+1)}{r!}$, for $r \geq 1$.

(a) Show that if n is an integer, then so is $C_r(n)$.

(b) Show that any polynomial $p(x)$ with rational coefficients can be expressed in the form

$b_k C_k(x) + b_{k-1} C_{k-1}(x) + \dots + b_0 C_0(x)$, where all the b_i's are rational and k is the degree of $p(x)$.

Show that all the b_i's are integers.

(c) Suppose that $p(x)$ is a polynomial with real coefficients such that whenever a is a rational number, then so is $p(a)$. Show that the coefficients of $p(x)$ are all rational.

48. Prove that $\quad$ **(a)** $\sum_{k=0}^{n} \left(C_k^n\right)^2 = \dfrac{(2n)!}{(n!)^2}$ $\quad$ **(b)** $C_k^n \dfrac{1}{n^k} \leq \dfrac{1}{k!}$ $\quad$ **(c)** $\left(1+\dfrac{1}{n}\right)^n < 3$

49. For any positive integers m, p such that $m \geq p-1$, let $G(m,p) = \dfrac{(1-x^m)(1-x^{m-1})\dots(1-x^{m-p+1})}{(1-x)(1-x^2)\dots(1-x^p)}$.

(a) Show that $G(m, p) = G(m, m-p)$ for $m > p$.

(b) Suppose $p \leq m-1$,

(i) Show that $\quad G(m, p+1) - G(m-1, p+1) = x^{m-p-1} G(m-1, p)$.

(ii) By putting $p+1, p+2, p+3, \dots$ for m in (i), or otherwise, show that

$G(m, p+1) = G(p, p) + x\,G(p+1, p) + x^2 G(p+2, p) + \dots + x^{m-p-1} G(m-1, p)$.

(c) Use induction on p to show that $G(m, p)$ is a polynomial in x for any positive integer m such that $m \geq p-1$.

50. Let $a_1, a_2, \dots, a_n$ be n distinct real numbers.

Let $f(a) = (x-a_1)(x-a_2)\dots(x-a_n)$ and $f'(x)$ the derivative of $f(x)$.

(a) Express $f'(a_i)$ in terms of $a_1, a_2, \dots, a_n$.

(b) Let $g(x)$ be a real polynomial of degree less than n.

(i) Show that there exist unique real numbers $A_1, A_2, \dots, A_n$ such that

$$g(x) = \sum_{i=0}^{n} A_i (x-1)\dots(x-a_{i-1})(x-a_{i+1})\dots(x-a_n) \qquad (*)$$

(ii) Using (i) or otherwise, show that if $f(x)$ is of degree less than $n-1$, then $\sum_{i=1}^{n} \dfrac{g(a_i)}{f'(a_i)} = 0$

(iii) By taking $a_i = i$ and suitable $f(x)$ in (ii), show that,

$$\sum_{i=1}^{n} (-1)^{n-i} \frac{i^m}{(i-1)!(n-i)!} = 0$$

(Given that $0! = 1$)